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FUNDAMENTAL FREQUENCIES OF TIMOSHENKO BEAMS MOUNTED ON PASTERNAK FOUNDATION

M. El-Mously

M.E. Department (312), University of Nevada at Reno, Reno, NV 89557, U.S.A

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1. INTRODUCTION

The notion of beam-on-elastic-foundation model has been extensively utilized by geotechnical, mechanical, pavement, and railway engineers for the analysis and design of foundation. The Bernoulli–Euler beam on Winkler foundation (BW) is often used to model the beam–foundation system. An improved theory for beams which accounts for the beam shearing-flexibility and rotary-inertia discarded in the classical theory, is usually referred to as the Timoshenko-beam theory [1]. A refined model for the soil–structure interaction which accounts for the foundation shearing-stiffness discarded in the Winkler model is the Pasternak model [2]. Figure 1 shows the layout and co-ordinate system of a uniform Timoshenko beam mounted on Pasternak foundation (TP).

Closed-form solutions for the modal characteristics of BW are well known [3]. These expressions afford a valuable insight into the system behaviour and its sensitivity to changes in any of the system parameters. Corresponding expressions for Timoshenko beams on Pasternak foundation are not available in the vast literature. Wang and Stephens [4] derived expressions for the frequency equations

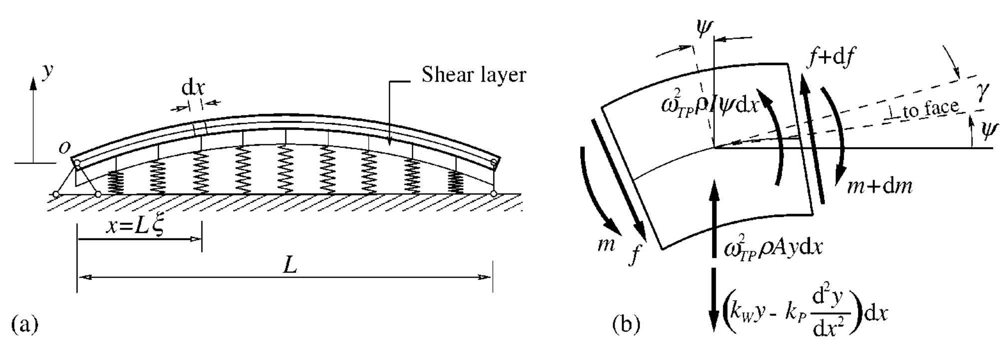

Figure 1. (a) A freely vibrating Timoshenko beam mounted on Pasternak foundation. The foundation is modelled as an infinite series of massless vertical springs of stiffness k_W per unit length, connected at top by a shearing layer of shearing stiffness k_P per unit length. (b) A small element of the beam of (a) showing forces and moments acting upon the element in their positive sense: f and m are the shearing force and the bending moment respectively.

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of finite Timoshenko-beams mounted on continuous Pasternak-foundation.These equations are highly transcendental, and no attempt has been made to solve them, even in simplest cases. The main aim of the present work is to derive explicit formulae for the fundamental natural frequencies for vibration of finite Timoshenko-beams mounted on finite Pasternak-foundation.

2. APPROXIMATE ANALYSIS

The Lagrangian $\Im$ of a TP can be expressed as

$$\Im_{\mathrm{TP}} = \frac{1}{2}\int_L (EI\psi'^2 + kAG\gamma^2 + k_W y^2 + k_P y'^2)\,\mathrm{d}x - \frac{1}{2}\int_L (\rho A y'^2 + \rho I \psi'^2)\,\mathrm{d}x, \quad (1)$$

where y is the lateral deflection of the beam reference-line; ψ and γ are the slopes due to bending and shearing, respectively; E and G; are the elasticity and rigidity moduli of the beam; respectively; k_W and k_P are the vertical and shearing stiffnesses of the foundation, respectively; ρ is the mass density of the beam; k is a shearing-deflection coefficient of the beam cross-section; A is the area of cross-section of the beam; I is the second moment of area of the beam cross-section around the principle axis normal to the plane of motion, primes and dots designate differentiation with respect to the axial co-ordinate x and time τ. The two integrals on the right-hand side of equation (1), respectively, correspond to the strain energy stored in the beam during deformation and the kinetic energy acquired by the beam.

It is convenient at this stage to define the following non-dimensional variables:

$$B = \omega L^2 \sqrt{\frac{EI}{\rho A}}, \qquad B_W^2 = \frac{k_W L^4}{EI}, \qquad R^2 = \frac{I}{AL^2}, \qquad S^2 = \frac{EI}{kAGL^2},$$

$$P^2 = \frac{k_P}{k_W L^2}, \quad (2)$$

where B is a frequency parameter for the beam or for the beam-foundation system, B_W is a frequency parameter for the Winkler foundation, R and S are parameters defining the beam rotary-inertia and shearing-flexibility, and P is a parameter defining the shearing stiffness of the foundation.

For $R = S = P = 0$, equation (1) reduces to that of a BW [3]. The beam shearing-flexibility and rotary-inertia allow the beam to deform with less strain energy; thus, reducing the natural frequencies of vibration by virtue of Rayleigh's principle. Whereas the shearing stiffness of the Pasternak foundation increases the strain energy; hence, increasing the natural frequencies. The present analysis is based on perturbing the BW model, such that the interaction between the foundation shearing-stiffness and both the beam shearing-flexibility and rotary-inertia is small enough to express the natural frequency of a TP as

$$B_{TP}^2 = B_T^2 + B_P^2, \quad (3)$$

where the subscripts T and P designate Timoshenko beams and Pasternak foundation respectively.

2.1. CORRECTIONS FOR THE ROTARY INERTIA AND SHEARING FLEXIBILITY OF THE BEAM

For a hinged–hinged Timoshenko beam (with no foundation, *i.e.*, $P = 0$), the non-dimensional frequency parameter B_T can be written [4] as

$$B_T = \frac{\pi^2}{\sqrt{1+\pi^2(R^2+S^2)}}\left[\sqrt{1+\frac{B_T^2}{\pi^4}(B_T RS)^2}\right] \cong \frac{\pi^2}{\sqrt{1+\pi^2(R^2+S^2)}}. \tag{4}$$

For small values of the coupling parameter $(B_T RS)^2$ compared to unity, the term between square brackets in equation (4) tends to unity. This approximation is analogous to ignoring terms involving fourth-order time-derivatives in the Timoshenko-beam theory, what is referred to as Love's equation; *cf.* Abramovich and Elishakoff [5]. This suggests postulating the following general approximate soultion for B_T:

$$B_T \cong \frac{B_B}{\sqrt{1 + C_S S^2 + C_R R^2}}, \tag{5}$$

where C_R and C_S are unknown constants. Application of Rayleigh's method, using the modal characteristics of the Bernoulli–Euler beam as trial modes, one finds

$$C_R = \int_0^1 y_B'^2 \,\mathrm{d}\xi \Big/ \int_0^1 y_B^2 \,\mathrm{d}\xi, \qquad C_S = \int_0^1 y_B'''^2 \,\mathrm{d}\xi \Big/ \int_0^1 y_B''^2 \,\mathrm{d}\xi, \tag{6,7}$$

where $\xi = x/L$ is a non-dimensional co-ordinate. The values of C_S and C_R are derived analytically for six different combinations of end restraints, including clamped (C), hinged (H), and free (F) ends; *cf.* Table 1. It should be mentioned that F/F and H/F beams allow for the rigid-body modes, which are associated with zero natural frequencies.

Table 1 shows that the constants C_S and C_R exhibit a special kind of symmetry. For H/H and C/F beams, the values of C_S and C_R are equal. Also, the values of C_S and C_R for C/C and C/H beams are equal to those associated with the modes next to the rigid-body modes of F/F and H/F beams, respectively, but with C_S and C_R interchanged. Indeed, this is due to the special kind of symmetry exhibited by the modal characteristics of the Bernoulli–Euler beams; see reference [3].

TABLE 1

Values of the constants B_B, C_S, C_R and C_P for various sets of end restraints

	C/C	C/F	C/H	H/H	H/F	F/F
B_B	22·4	3·50	15·4	π^2	0 (15·4)*	0 (22·4)*
C_S	$5{\cdot}01\,\pi^2$	$0{\cdot}47\,\pi^2$	$2{\cdot}76\,\pi^2$	π^2	0 $(1{\cdot}17\,\pi^2)$*	0 $(1{\cdot}25\,\pi^2)$*
C_R	$1{\cdot}25\,\pi^2$	$0{\cdot}47\,\pi^2$	$1{\cdot}17\,\pi^2$	π^2	0 $(2{\cdot}76\,\pi^2)$*	0 $(5{\cdot}01\,\pi^2)$*
C_P	$1{\cdot}25\,\pi^2$	$0{\cdot}47\,\pi^2$	$1{\cdot}17\,\pi^2$	π^2	3 $(2{\cdot}76\,\pi^2)$*	0 $(5{\cdot}01\,\pi^2)$*

* Values between parentheses (H/F & F/F) are associated with the next-higher order modes.

2.2. CORRECTIONS FOR THE SHEARING STIFFNESS OF THE FOUNDATION

Using Rayleigh's method, the increase in the natural frequency of the Winkler foundation B_W due to the shearing stiffness of the Pasternak foundation can be expressed as

$$B_P = B_W\sqrt{1 + C_P P^2}, \qquad C_P = \int_0^1 y_B'^2\,\mathrm{d}\xi \bigg/ \int_0^1 y_B^2\,\mathrm{d}\xi. \tag{8,9}$$

Equation (9) shows that the values of C_P are equal to those of C_W, except for the rigid-body modes; see Table 1.

3. COMPARATIVE STUDY

The fundamental natural frequency for vibration of finite Timoshenko-beams mounted on finite Pasternak-foundation can thus be expressed as

$$B_{TP}^2 = B_B^2(1 + C_S S^2 + C_R R^2)^{-1} + B_W^2(1 + C_P P^2). \tag{10}$$

A comparative study was conducted between the results of equation (10) and the numerical solutions of the Hamilton-derived equations of motion of the system described by equation (1) for: (1) $B_W/B_B = 0$, which corresponds to a Timoshenko beam without foundation, and (2) $B_W/B_B = 1$, which corresponds to a Timoshenko beam mounted on a Pasternak foundation whose vertical stiffness equals the flexural stiffness of the beam.

Figure 2 shows plots of the variation of the frequency ratio B_T/B_B ($= B_{TP}/B_{BW}$ for $P = 0$) against the non-dimensional shearing parameter S of a TP having $B_W/B_B = 0$. Two values of R/S are considered: (a) $R/\mathrm{S} = 0$, and (b) $R/S = 0{\cdot}5$. The figure shows that the results of equation (10) agree well with the exact solution of equation (1), with less than 5% relative error.

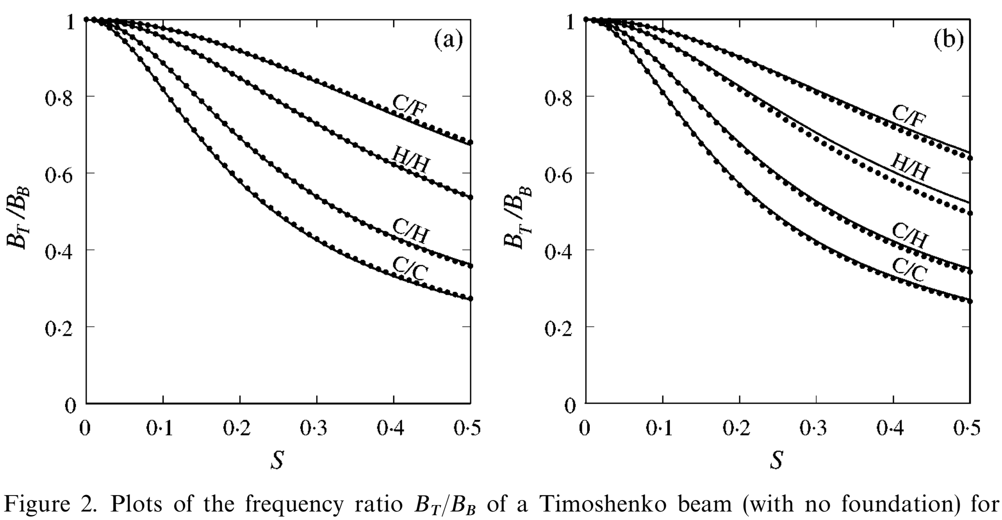

Figure 2. Plots of the frequency ratio B_T/B_B of a Timoshenko beam (with no foundation) for different sets of end restraints. Solid and dotted curves correspond, respectively, to the exact solution, and the approximate explicit formulae (10) for (a) $R/S = 0$, and (b) $R/S = 0{\cdot}5$.

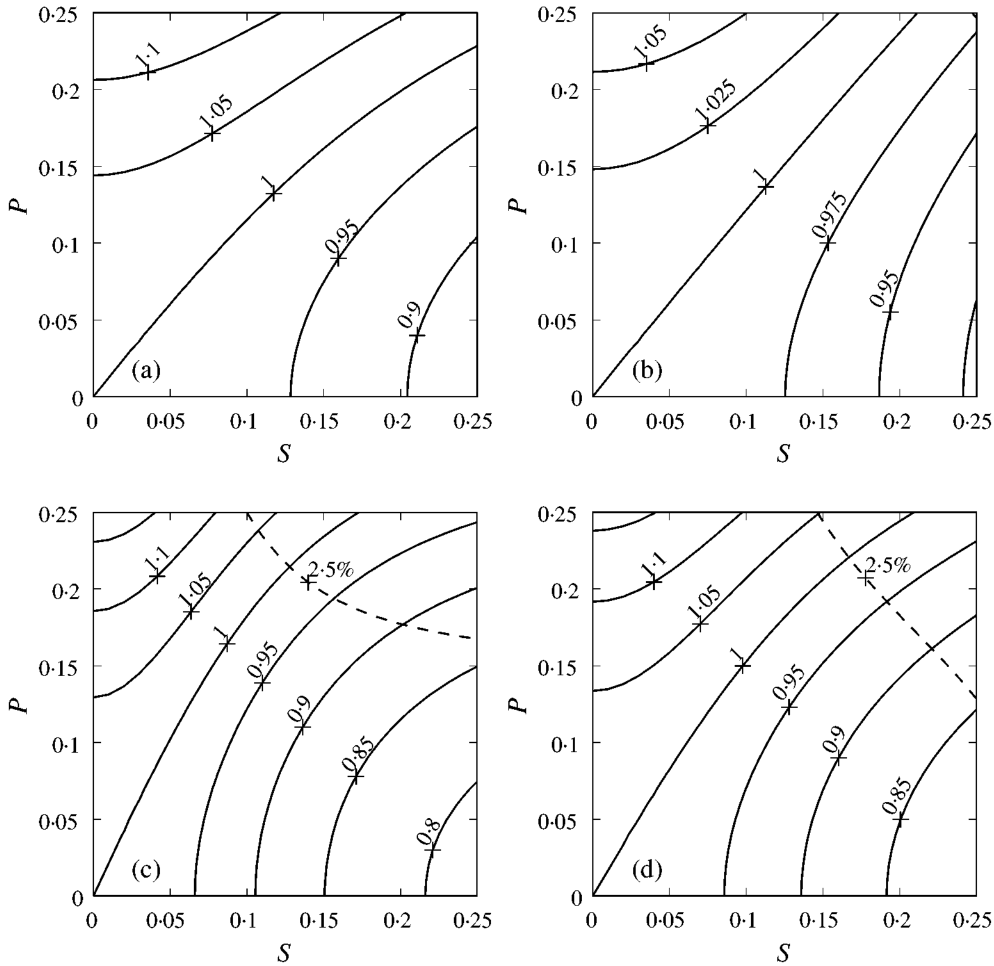

Figure 3. Contour plots of the frequency ratio B_{TP}/B_{BW} of a Timoshenko beam mounted on Pasternak foundation, having different sets of end restraints: (a) H/H, (b) C/F, (c) C/C, (d) C/H, all having $B_W/B_B = 1$. Broken curves are contours of the percentage of error in estimating the frequency ratio B_{TP}/B_{BW} using equation (10) compared with the exact analytical solution. For all curves, the percentage of relative errors are less than 5%.

Figure 3 shows contour plots of the frequency ratio B_{TP}/B_{BW} for different sets of end restraints, all having $B_W/B_B = 1$ and $R/S = 0{\cdot}5$. Broken curves are contours of the percentage of relative error incurred by equation (10) compared with the exact solution. The range of S and P considered in Figure 3 is chosen such that $(B_{TP}RS)^2 < 1$. which is a cut-off for higher spectra of frequencies; see reference [4]. Figure 3 shows that the results obtained by equation (10) agree well with the exact solutions over the domain of geometry considered.

4. EFFECT OF FOUNDATION CONTINUITY

Equation (10) is derived for finite beams mounted on finite foundations. They are also applicable for finite beams mounted on continuous foundations, except for beams having free ends. In such a case, the deformation of the free ends incurs deformation in the foundation away from the beam, which adds to the strain energy stored in the system, and subsequently to the natural frequencies of vibration. The contribution to the fundamental frequency due to the continuity of the foundation is obtained by using Rayleigh's principle, where the exponentially decaying deflection of the unloaded surface of foundation is obtained from the equation of motion of the Pasternak foundation [2]. In the cases of C/F, H/F, and F/F beams, the contributions to B_{TP} (equation (10)) due to the continuity to the foundation are found to be $4(B_W)^2P$, $3(B_W)^2P$, and $2(B_W)^2P$ respectively.

5. CONCLUSIONS

Approximate explicit formulae have been derived for the fundamental natural frequency for vibration of Timoshenko beams mounted on Pasternak foundation. These formulae agree well with the exact solution of the beam–foundation system.

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